# ON A FORM OF STEADY MOTION IN HYDRODYNAMICS

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We shall find the characteristics of the established wave arising in an unstable system which is described by the one-dimensional hydrodynamic equation.

We shall prove that for slight supercriticality $\lambda - \lambda_*$ the amplitude of the $n$th harmonics of the wave are proportional to $(\lambda - \lambda_*)^{1/2 n}$.

The equations of hydrodynamics usually have an equilibrium (time independent) solution. To a stable equilibrium solution there corresponds a laminar motion of the medium.

The equilibrium solution depends upon external parameters; as the parameters pass through critical values the equilibrium solution becomes unstable. Nonlinear effects (the interaction of waves with different wave numbers growing in the unstable system) will limit the growth of the fluctuations; as a result the system achieves a steady state. In such a steady state either there is a continuous spectrum of waves (turbulent motion)[1 and 2], or else there exists one wave with a definite frequency and wave vector (for example, the strata [3] in a gaseous discharge and the "diffusion" oscillations in a strong magnetic field [4]).

Below we shall consider the second case. The steady solution will be calculated; the stability of the steady solution will not be considered. It is assumed that the starting equations describe the system sufficiently well, and that it is known from experiment that the steady state of the system is of the second form.

**1.** Let the unknown functions $X^j(t,x)$ satisfy the autonomous equations (not containing explicitly $t$ or $x$)

$$F_0^i\left(X^j, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = 0 \qquad (i, j = 1, \ldots, n) \tag{1.1}$$

In Equations (1.1) there occur external parameters, the totality of which will be denoted by $\lambda$.

The equilibrium solution $X^j$, not depending upon $t$ or $x$, is found from Equations

$$F^i \equiv F^i(X^i, 0, 0) = 0 \tag{1.2}$$

In the study of the stability of the equilibrium state let us set

$$X^j = X^j + X_1 e^{i\theta}, \qquad \theta = \omega t - kx$$

After linearizing Equations (1.1) with respect to the perturbations $X_1$ we obtain the system of algebraic equations

$$a_j^i X_1^j = 0 \tag{1.3}$$

Here and henceforth, two identical indices, one of which is a subscript and the other a superscript, will imply summation from 1 to $n$. In Equations (1.3) the coefficients $a$ are known functions of $\omega$, $k$, $\lambda$ and $X(\lambda)$.

Nontrivial solutions of the equations exist if the dispersion relation is fulfilled

$$D \equiv |a_j^i| = 0 \tag{1.4}$$

From the dispersion relation we can determine the complex frequency $\omega = \Omega - i\gamma$ of each mode as a function of the external parameters and of the (real) wave number $k$. Let us assume that we have been able to do this and have established that for certain critical values of the external parameters $\lambda_*$ the maximal (as a function of the wave number $k$) increment of one of the modes is equal to zero, whilst the increments of the other modes are everywhere negative. If we slightly change the defined direction of some of the external parameters, then the increment of the "critical" mode becomes positive in a certain interval of values of $k$, whilst the increments of the other modes remain negative (in what follows the root of the dispersion relation $D(\omega) = 0$, characterizing the "critical" mode, will be denoted by $\omega_* = \Omega_* - i\gamma_*$). It is natural to expect that the steady wave is the result of the development of oscillations of the unstable mode.

Let us suppose that with steady increase of supercriticality $\lambda - \lambda_*$ the amplitude of the established wave steadily increases from zero (such behavior of the resulting wave with change of the parameter will be called "smooth"). Then for weak supercriticality many properties of the wave must be determined by the properties of the unstable mode; in particular, the wave number and the frequency of the wave must be close to the values $k_0$ and $\omega_0$, determined in the linear theory from Equations

$$\partial\gamma_* / \partial k = 0, \quad \omega_0 = \Omega_*(k_0) \tag{1.5}$$

**2.** Assuming that the maximal increment $\gamma_0$ of the unstable mode is positive, but sufficiently small (i.e. the supercriticality is weak), let us proceed to the calculation of the quantities characterizing the established wave.

The established solution obviously has the form

$$X^j = \sum_{\nu=-\infty}^{\infty} X_\nu^j e^{i\nu\theta}, \qquad \theta = \omega t - kx \tag{2.1}$$

where the quantities $X_{-\nu}$ and $X_\nu$ are complex conjugates: $X_{-\nu}^j = (X_\nu^j)^*$; this follows from the fact that the quantities $X^j$ are real.

Substituting (2.1) in (1.1), let us multiply the result by $e^{-i\mu\theta}$ and integrate with respect to $\theta$ from 0 to $2\pi$. As a result we obtain

$$\Phi_\mu{}^i \equiv \int_0^{2\pi} F^i e^{-i\mu\theta} \frac{d\theta}{2\pi} = 0 \tag{2.2}$$

Equations (2.2) form an infinite system of algebraic equations for the infinite number of unknowns $X_\nu$. To solve this system in the general case is impossible; even the integration in (2.2) often cannot be achieved in an explicit form.

Now let us make use of the circumstance that for weak supercriticality the amplitude of the wave is small if the oscillations arise "smoothly". Then the integrals in (2.2) can be calculated if we write down Expressions (2.1) in the form $X^j = X_0{}^j + X_\sim{}^j$ and then expand the functions $F$ with respect to the small quantities $X_\sim$ (it is necessary to emphasise that $X_0 \neq X$).

To find the first harmonics $X_1$, which are the greatest, it is sufficient to calculate only the functions $\Phi_\mu$ with $\mu = 0,1,2$, retaining only the quadratic and cubic terms in $X_\sim$ and taking into account in the expressions for $X_\sim$ only the first and second harmonics

$$\Phi_0{}^i \equiv F_0{}^i(X_0{}^j) + c_{jk}^i X_1{}^j X_{-1}{}^k = 0 \tag{2.3}$$

$$\Phi_1{}^i \equiv a_j{}^i X_1{}^j + d_{sl}^j X_2{}^s X_{-1}{}^l + b_{jkl}^i X_1{}^j X_1{}^k X_{-1}{}^l = 0 \tag{2.4}$$

$$\Phi_2{}^i \equiv f_j{}^i X_2{}^j + g_{jk}^i X_1{}^j X_1{}^k = 0 \tag{2.5}$$

In these equations the coefficients $a, b, \ldots, g$ are known functions of $\omega$, $k$ and $X_0$; the tensors $b$, $g$ can be assumed symmetric with respect to $j$ and $k$.

We note that the sum of the subscript indices of the symbols $X$, appearing in any term of the expressions $\Phi_\mu{}^i$, is always equal to $\mu$.

Writing $X_0{}^j = X^j + \delta X^j$ and remembering that $F_0{}^i(X^j) = 0$, we obtain from Equation (2.3)

$$(\partial F_0{}^i / \partial X_0{}^j)_X \, \delta X^j + c_{jk}^i X_1{}^j X_{-1}{}^k = 0 \tag{2.6}$$

From the last expression it is clear that $\delta X \sim (X_\sim)^2$, therefore in the quantities $b, \ldots, g$ we must set $X_0 = X$, in order not to exceed the accuracy of Equations (2.3) to (2.5); for the same reason in the expansion of the quantities $a$ with resect to $\delta X$ we must retain only linear terms

$$a_j{}^i(X_0{}^m) = a_j{}^i(X^m) + (\partial a_j{}^i / \partial X_0{}^m)_X \, \delta X^m \tag{2.7}$$

According to the known rules of linear algebra let us solve the system of equations (2.5) with respect to $X_2$ and the system (2.6) with resect to $\delta X$

$$X_2{}^s = G_{jk}^s X_1{}^j X_1{}^k, \qquad \delta X^m = C_{kl}^m X_1{}^k X_{-1}{}^l \tag{2.8}$$

Substituting Expressions (2.7) and (2.8) in Equation (2.4), we obtain

$$\Phi_1^{\,i} \equiv X_1^{\,j}\,(a_j^{\,i} + P_{jkl}^{\,i} X_1^{\,k} X_{-1}^{\,l}) = 0 \tag{2.9}$$

where

$$P_{jkl}^{\,i} = (\partial a_j^{\,i} / \partial X_0^{\,m})_{\rm X}\, C_{kl}^{\,m} + d_{sl}^{\,i} G_{jk}^{\,s} + b_{jkl}^{\,i}$$

The system of linear homogeneous equations (2.9) in $X_1^{\,j}$ has a nontrivial solution if

$$\Delta \equiv |\, a_j^{\,i} + P_{jkl}^{\,i} X_1^{\,k} X_{-1}^{\,l} \,| = 0 \tag{2.10}$$

When the nonlinear relation (2.10) is fulfilled, the solution of Equation (2.9) is

$$X_1^{\,j} = Q \Delta_p^{\,j} \tag{2.11}$$

Here $Q$ is a constant of proportioality, whilst $\Delta_p^{\,j}$ is the co-factor of the element $A_j^{\,p}$, standing in the determinant $\Delta$ at the intersection of the $j$th column and the $p$th row (the number of the row $p$ is arbitrary).

Now we note that, according to the assumption of "smooth" excitation, $X_1^j \to 0$ when $\lambda \to \lambda_*$. According to (2.10), the determinant $\Delta \to D$ as $X_1 \to 0$, and accordingly $\Delta_p^{\,j} \to D_p^{\,j} \neq 0$, so that from Equation (2.11) we find that $Q \to 0$ as $\lambda \to \lambda_*$.

In connection with this we neglect the quadratic terms in the expressions for $\Delta_p^{\,j}$, and Equations (2.11) take the form

$$X_1^{\,j} = Q D_p^{\,j} \tag{2.12}$$

From the last equation it is clear that in the steady wave with weak supercriticality the phsse relations between the quantities $X_1^{\,j}$ are determined by the linear theory; the quantity $Q$ is none other than the normal coordinate of the oscillation in the unstable mode.

To determine $Q$ let us substitute Expressions (2.12) for $X_1$ in the nonlinear dispersion relation (2.10) and expand $\Delta$ in series with respect to the small quantity $q = QQ^*$, retaining only terms linear in $q$ (retention of higher degrees would be exceeding the accuracy of the starting equations (2.3) to (2.5))

$$\Delta \equiv D + Pq = 0, \qquad P = D_i^{\,j} P_{jkl}^{\,i} D_p^{\,k} (D_p^{\,l})^*, \qquad q = QQ^* \tag{2.13}$$

From Equation (2.13) let us determine the complex frequency $\omega$ as a function of the wave number $k$ and the small positive parameter $q$; the solution will be sought in the form $\omega = \omega_* + \delta\omega$ where $\omega_*$ is determined in the linear theory from Equation $D(\omega_*) = 0$, whilst $\delta\omega \sim q$. From Equation (2.13) we obtain

$$\delta\omega = -\,qS, \qquad S = \left(\frac{P}{\partial D / \partial \omega}\right)_{\omega_*} \tag{2.14}$$

and the expression for the frequency takes the form

$$\omega = (\Omega_* - qS^{(r)}) - i\,(\gamma_* + qS^{(i)}) \tag{2.15}$$

Here and in what follows the superscript

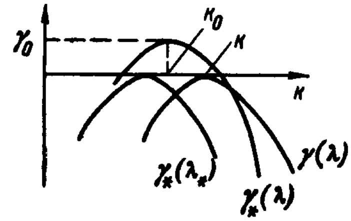


Fig. 1

indices in papentheses $(r)$ and $(i)$ denote the real and imaginary parts of the corresponding quantity.

Now let us suppose that the wave that becomes established in the system is of such amplitude that the maximum nonlinear increment $\gamma \equiv \gamma_* + qS^{(i)}$ as a function of the wave number $k$ vanishes (Fig.1); the value $k$, for which $\gamma = 0$, is in fact the wave number of the established wave, whilst the corresponding value $\omega(k)$ is the frequency of the wave. This supposition arises in connection with the fact that the value of the parameter $q$, for which the maximum nonlinear increment is equal to zero, is a qualitatively unique choice from among the other values of $q$.

According to supposition made, the quantities $q$ and $k$ of the established wave must satisfy Equations

$$\gamma \equiv \gamma_* + qS^{(i)} = 0, \qquad \partial\gamma / \partial k = 0 \tag{2.16}$$

The wave number will be sought in the form $k = k_0 + \delta k$, where $\delta k \sim q$ and the quantity $k_0$ is determined in the linear theory from Equation $\partial\gamma_*/\partial k = 0$; then Equations (2.16) take the form

$$\gamma_0 + qS_0^{(i)} = 0, \qquad \gamma_0''\delta k + q\,(S_0')^{(i)} = 0 \tag{2.17}$$

Here the primes denote differentiation with respect to $k$, whilst the subscript 0 shows that the corresponding quantity is taken when $k = k_0$.

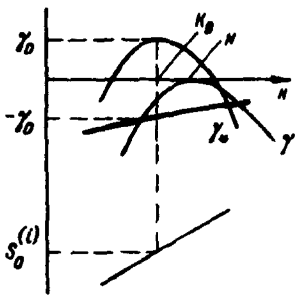

Fig. 2

From the first equation we find that

$$q = -\gamma_0 / S_0^{(i)}.$$

Since $q \sim \gamma_0$, then in Expression (2.14) for $S$ we must set $\omega_*(k_0) = \Omega_*(k_0) = \omega_0$ (the retention in the expression for $S$ of terms $\sim \gamma_0$ would exceed the accuracy). Since $\gamma_0 > 0$ and, obviously, also $q > 0$, then the solution exists if $S_0^{(i)} < 0$. The last inequality is a sufficient condition for a "smooth" onset of oscillations.

The deviation $\delta k$ is determined from the second of Equations (2.17), whilst the frequency $\omega$ comes from Equation (2.15); accordingly, we can write down

$$q = -\gamma_0 / S_0^{(i)}, \quad \text{if} \quad S_0^{(i)} < 0 \tag{2.18}$$

$$\delta k = -q\,(S_0')^{(i)} / \gamma_0'' \tag{2.19}$$

$$\omega = \omega_0 + (\partial\Omega_* / \partial k)_0\, \delta k - qS_0^{(r)} \tag{2.20}$$

The expressions for $q$ and $\delta k$ obtained are such that for the fulfilment of the condition $0 < \gamma_0 \leqslant -S_0^{(i)}$ the linear increment would be depicted by the parabola $\gamma_* = \gamma_0 + {}^1/_2\gamma_0''(k - k_0)^2$, whilst the function $S^{(i)}$ would be the straight line $S^{(i)} = S_0^{(i)} + (S_0')^{(i)}(k - k_0)$ (Fig.2; the straight line

$qS^{(i)}$ is shown by the bold line) (*).

**3.** The first harmonics $X_1$ are determined by Formula (2.12), in which the cofactors $D_p{}^j$ are evaluated when $\omega = \omega_0$ and $k = k_0$ and the quantity $Q$ is determined by Equation (2.18) (Equation 2.18 determines only the modulus of the complex amplitude, and therefore the steady solution turns out to be determined only to the extent that its phase remains arbitrary).

If the first harmonics are known, then the second harmonics are found from Equations (2.5), which are linear in $X_2$, and have the form $X_2^j = H_2{}^j Q^2$, where the quantities $H_2$ are expessed in terms of the coefficients $f$, $g$ in the expression for $\Phi_2$ and the already known quantities $H_1{}^j \equiv D_p{}^j$ (according to Expressions (2.8) and (2.12), $H_2{}^s = G_{jk}^s D_p{}^j D_p{}^k$).

In a similar way we can find the higher harmonics if all the harmonics of lower orders are known. For example, the third harmonics $X_3$ are expressed in terms of $X_1$, $X_2$ from Equations $\Phi_3 = 0$, in which enter terms of the form $X_3$, $X_2X_1$, $X_1X_1X_1$ (these terms arise, respectively, from terms of order $X_\sim$, $X_\sim{}^2$, $X_\sim{}^3$ in the expansions of the starting equations with respect to the quantity $X_\sim$). Having solved Equations $\Phi_3 = 0$, for $X_3$ we obtain $X_3{}^j = H_3{}^j Q^3$, where the quantities $H_3$ are expressed in terms of the already determined quantities $H_1$, $H_2$ and the coefficients in the expressions for $\Phi_3$.

In general, the harmonics $X_\nu$ will be determined from Equations

$$\Phi_\nu{}^i \equiv (h_\nu)_j{}^i X_\nu{}^j + \sum (h_{\nu_1\nu_2\ldots\nu_s})_{j_1j_2\ldots j_s}^{i} X_{\nu_1}{}^{j_1} X_{\nu_2}{}^{j_2} \ldots X_{\nu_s}^{j_s} = 0 \qquad (3.1)$$

Here the sum is taken over all possible combinations of the members $\nu_i$ satisfying the conditions : $\nu_1 + \ldots + \nu_s = \nu$, $1 \leqslant \nu_1, \ldots, \nu_s < \nu$; the coefficients $h$ are taken at $k = k_0$, $\omega = \omega_0$, $X_0 = X$. We notice that in Equations (3.1) there are contained terms of the form $(X_1)^\nu$, arising from terms in the expansion of order $(X_\sim)^\nu$; the terms of the expansion with respect to $X_\sim$ of higher order do not contribute to Equations (3.1).

It was noticed above that $X_\nu{}^j = Q^\nu H_\nu{}^j$ if $\nu \leqslant 3$. Let us assume that this relation is true for all values of $\nu$, including $\nu = \mu - 1$, and the coefficients $H_1, \ldots, H_{\mu-1}$ already found; then with the help of Equation (3.1) it is not difficult to see (taking account of the condition $\nu_1 + \ldots + \nu_s = \nu$) that it is true also for $\nu = \mu$ (and consequently for all $\nu$). Moreover the coefficients $H_\mu$ are expressed in terms of the already determined coefficients $H_1, \ldots, H_{\mu-1}$ and the quantities $h$ in Equations (3.1) with $\nu = \mu$. By means of the estimate $|X_\nu| \sim |Q|^{|\nu|}$ it can be established that the ratio of the omitted terms in the expressions for $\Phi$ to those retained has the order of smallness $q$ or higher than $q$; accordingly we can write

*) If the unstable mode is aperiodic, i.e. $\omega_0 = 0$ and moreover $(\partial \Omega_*/\partial K) = S_0{}^{(r)} = 0$, then a motion which is independent of time becomes established in the system.

$$X_\nu^j = Q^\nu [H_\nu^j + O(q)], \qquad \nu > 0; \qquad X_{-\nu}^j = (X_\nu^j)^* \tag{3.2}$$

Thus, the $\nu$th harmonics of the established wave are proportional to $\gamma_0^{1/2\nu}$.

4. We shall now express in explicit form the amplitudes of the harmonics of the established wave in terms of the supercriticality $\varepsilon = \lambda - \lambda_*$ (here $\lambda_*$ is a certain fixed point of the surface of critical parameters; this surface is given in the space of the parameters by the equation $\gamma_0(\lambda) = 0$).

The functions $F$ in (1.1) are analytic functions of their arguments and the external parameters $\lambda$, therefore the quantities $X$, $\gamma_0$, $k_0$, $\omega_0$, and hence all the quantities characterizing the wave, are analytic functions of $\lambda$. For sufficiently weak supercriticality $\varepsilon$ we can therefore write down (remembering that $\gamma_0 = 0$ when $\lambda = \lambda_*$)

$$\gamma_0 = (d\gamma_0 / d\lambda)_* \, \varepsilon \tag{4.1}$$

This expression (*) gives the possibility of determining how to change the external parameters in order to secure that $\gamma_0 > 0$.

Let us substitute Expression (4.1) for $\gamma_0$ in Formulas (2.18) and (3.2) for $Q$ and $X_\nu$, setting $\lambda = \lambda_*$ in the expressions for $S$ and $H$ (it is assumed that the quantities $\omega_0(\lambda)$, $k_0(\lambda)$, $X(\lambda)$ have already been substituted in the expressions for $\gamma_0$, $S$, $H$); as a result we find (**) (under the condition $S_0^{(i)}(\lambda_*) < 0$), that $X_\nu \sim \varepsilon^{1/2\nu}$.

In the expression for the frequency it is necessary to make the substitution $\omega_0(\lambda) = \omega_0(\lambda_*) + (d\omega_0/d\lambda)_*\varepsilon$; the change of frequency $(d\omega_0/d\lambda)_*\varepsilon$ is not connected with the presence of oscillations and is valid for a variation $\varepsilon$ in any direction. A similar substitution is effected for $k_0(\lambda)$.

Hence, taking account of the fact that $\gamma$ is real, we find that

$$D^{(r)}\frac{\partial D^{(r)}}{\partial\omega} + D^{(i)}\frac{\partial D^{(i)}}{\partial\omega} = 0, \qquad \gamma = \left|\frac{\partial D}{\partial\omega}\right|^{-2} \begin{vmatrix} \partial D^{(r)}/\partial\omega & \partial D^{(i)}/\partial\omega \\ D^{(r)} & D^{(i)} \end{vmatrix} \tag{5.1}$$

5. The equation $D(\omega) = 0$ is often difficult to solve and it is then necessary to find $\omega$ approximately.

Let us expand $D = D^{(r)} + iD^{(i)}$ with respect to the small quantity $\gamma$ and restrict ourselves to the linear term

$$D(\Omega - i\gamma) \equiv D(\Omega) - i\gamma\,(\partial D/\partial\omega)_\Omega = 0$$

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*) The quantity $\varepsilon$ is to be understood as a vector, whilst derivatives with respect to $\lambda$ are tensors; written out in detail, Equation (4.1), for example, has the form

$$\gamma_0 = \left(\frac{\partial\gamma_0}{\partial\lambda_i}\right)_* \varepsilon_i + \frac{1}{2}\left(\frac{d^2\gamma_0}{d\lambda_i d\lambda_j}\right)_* \varepsilon_i\varepsilon_j + \ldots$$

**) From the relation $X_\nu \sim (\sqrt{\varepsilon})^\nu$ it follows that we can seek the steady solution directly in the form of series with respect to $\sqrt{\varepsilon}$.

The first equation determines the frequency $\Omega$ ; substituting the value found for $\Omega$ in the second equation (5.1), we obtain $\gamma$ (*).

When $\lambda = \lambda_*$ we have $\gamma_0 = 0$ ; hence also from (5.1) it follows that $\lambda_*$, $k_0(\lambda_*)$, $\omega_0(\lambda_*)$ satisfy Equations (**)

$$D^{(r)} = 0, \qquad D^{(i)} = 0, \qquad \frac{\partial (D^{(r)}, D^{(i)})}{\partial (\omega, k)} = 0 \tag{5.2}$$

From (5.1) and (5.2) we obtain

$$\left(\frac{d\gamma_0}{d\lambda}\right)_* = \left|\frac{\partial D}{\partial \omega}\right|^{-2} \begin{vmatrix} \partial D^{(r)} / \partial \omega & \partial D^{(i)} / \partial \omega \\ D_\lambda{}^{(r)} & D_\lambda{}^{(i)} \end{vmatrix} \tag{5.3}$$

where

$$D_\lambda \equiv \left(\frac{\partial}{\partial \lambda} + \frac{dX}{\partial \lambda}\frac{\partial}{\partial X}\right) D$$

Now from (2.14), (2.18), (4.1) and (5.3) we have

$$q = -\varepsilon \begin{vmatrix} \partial D^{(r)} / \partial \omega & \partial D^{(i)} / \partial \omega \\ D_\lambda{}^{(r)} & D_\lambda{}^{(i)} \end{vmatrix} \cdot \begin{vmatrix} \partial D^{(r)} / \partial \omega & \partial D^{(i)} / \partial \omega \\ P^{(r)} & P^{(i)} \end{vmatrix}^{-1} \tag{5.4}$$

The quantities $\lambda$, $\omega$, $k$ in the right-hand terms of (5.3) and (5.4) are equal, respectively, to $\lambda_*$, $\omega_0(\lambda_*)$, $k_0(\lambda_*)$ and are found from Equations (5.2) (***).

**6.** Let us consider the question of the nonunique choice of the tensors $b$, $g$, and consequently also $P$ (Equations (2.4), (2.5) and (2.10)). The nonuniqueness is connected with the fact that, for example, the actual sum $g_{jk}^i X_1^j X_1^k$ (which is obtained in the calculation of $\Phi_2^i$) uniquely determines only the tensor $g$ symmetric with respect to $j$ and $k$ (if we add to it an arbitrary tensor which is antisymmetric with respect to $j$ and $k$, the sum is not changed). In order that the results obtained above should not depend upon the choice of the tensors $b$, $g$, $P$, it is necessary, as follows from (2.9), (2.13) and (2.14), that the equation $D_i{}^j D_p{}^k = D_i{}^k D_p{}^j$ should be satisfied under the condition $\omega = \omega_*$. The equation will indeed be satisfied, since $D(\omega_*) = 0$ and for an arbitrary determinant $D$ the following identity holds [6]:

$$\begin{vmatrix} D_i{}^j & D_i{}^k \\ D_p{}^j & D_p{}^k \end{vmatrix} = D D_{ip}^{jk} (-1)^N$$

where $N$ is the parity of the permutation $\binom{jk}{ip}$.

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*) It can happen that in the expression for $\Omega$ there appear terms $\sim \gamma^2$ (for example, for the ordinary equation $x'' - hx' + x = 0$ we obtain $D = 1 - \omega^2 - i\omega h$ and according to (5.1) we have $\Omega^2 = 1 - \frac{1}{4}h^2$, $\gamma = \frac{1}{2}h$); these must be discarded, in order not to exceed the accuracy of Equations (5.1).

**) The critical parameters (if they exist) for the other modes can likewise be found from (5.2).

***) The derivatives $(d\omega_0/d\lambda)$, $(dk_0/d\lambda)$ when $\lambda = \lambda_*$ also are expressed in terms of $D$ (the corresponding expressions are not reproduced for reasons of space economy). If Equations (1.1) are ordinary, then from the first equation (5.1) we obtain, taking account of (5.2)

$$(d\omega_0 / d\lambda)_* = |\partial D / \partial \omega|^{-2} [D_\lambda{}^{(r)} (\partial D^{(r)} / \partial \omega) + D_\lambda{}^{(i)} (\partial D^{(i)} / \partial \omega)]$$

**7.** The condition of "smooth" excitation is $S_0^{(i)}(\lambda_*) < 0$; if it is fulfilled, then for sufficiently slight supercriticality the amplitude of the wave is small, and its shape is close to sinusoidal, since $X_\nu \sim \varepsilon^{1/2\nu}$.

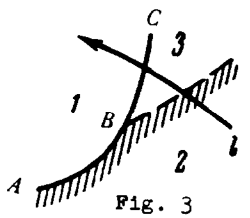

Fig. 3

If $S_0^{(i)}(\lambda_*) > 0$, then the oscillations arise "abruptly". Let us suppose for simplicity that the state of the system is determined by two parameters (Fig.3). Suppose that $\gamma_0 = 0$ on the curve $AC$ ($\gamma_0 > 0$ in the region 1) and $S_0^{(i)} > 0$ on the segment $BC$. If the parameters slowly change along the curve $l$, then jumps in amplitude of the established wave will take place at the intersection of the curve $BC$ and a certain curve (the broken line in Fig.3) in the region where $\gamma_0 < 0$. When the point $\lambda$ lies in region 3, and oscillations are initially absent, then they can be excited by an external perturbation if the amplitude of the perturbation $Q$ exceeds a certain threshhold value $Q_*$. From Equation $\gamma \equiv \gamma_0 + S^{(i)}q + \ldots = 0$ it follows that $|Q_*|^2 \to -\gamma_0 / S_0^{(i)}$ when $\gamma_0 \to -0$.

The type of hysteresis of the jump in amplitude can be explained in the following way. In the region 3 the increment $\gamma_0 < 0$ and therefore the fluctuating perturbations do not grow. After crossing into region 1 we have $\gamma_0 = +0$, but the nonlinear increment $\gamma = \gamma_0 + S_0^{(i)}q$ increases with the growth of $q$, since $S_0^{(i)} > 0$. Growth of the perturbations may be limited by nonlinear terms of higher order; since they are taken into account, the increment $\gamma$ will contain terms which are nonlinear with respect to $q$ and therefore there may exist positive solutions of Equation $\gamma(q) = 0$, even if $\gamma_0 \leqslant 0$.

On deeper penetration into the region $\gamma_0 > 0$ (region 1 in Fig.3) the amplitude of the wave increases, and the oscillations assume a relaxational character; moreover, there is the possibility of jumps in the wave amplitude of a hysteresis type (examples of such jumps for systems described by the regular equations are to be found in [5]). In addition, other modes may become excited; because of nonlinearity of the system oscillations of different modes may synchronize. Certain of the newly excited modes may have negative increments $\gamma_0$. Such modes are excited "abruptly", and moreover the "external" perturbation for them is the wave excited at a smaller supercriticality. As usual, in the case of "abrupt" excitation, the region of emergence of oscillations of such modes must be wider than the region of origin.

If the oscillations appear smoothly, then we can always choose such a small supercriticality that the other modes are not excited (not only in the linear approximation, but even in the presence of an extablished oscillation of an unstable mode). This case was in fact considered above quantitatively (*).

**8.** All the results obtained above are valid for a system with a finite number of degrees of freedom, when Equations (1.1) are ordinary.

Since Equations (1.1) are real and are polynomials with respect to the differential operators, the coefficients of the equation $D = 0$ which is algebraic with respect to $(i\omega)$ are polynomials with respect to $(-ik)$ with real coefficients. On replacing $k$ by $(-k)$ these polynomials become complex conjugates, and therefore the following relationship is true:

$$\omega(-k) = -\omega^*(k) \qquad (8.1)$$

from which it follows that the increment $\gamma(k)$ is an even function, whilst the frequency $\Omega$ is an odd function, and, generally speaking, discontinuous at $k = 0$.(**)

*) The cited qualitative chart of the dependence of oscillations on the supercriticality clearly shows, for example, many lines of strata in gaseous discharges.

**) The relation (8.1) reflects the fact that the direction of propagation of a wave is determined by the physical properties of the system, but not by choice of direction of the coordinate axis of reference.

If $\partial / \partial x \equiv 0$, $k = 0$, then to each oscillatory mode there corresond two roots $\omega_1$ and $\omega_2$ such that $\omega_2 = -\omega_1^*$.

It can be shown that if the condition of "smooth" excitation is satisfied, then for sufficiently small supercriticality the steady solution of the ordinary equations is stable (in particular, the solution $\chi = X$ when $\lambda = \lambda_*$ is stable).

**9.** In conclusion let us notice the character of the nonlinear effects taken into account.

The first term in Expression (2.9) for $P_{kl}^1$ is related to the effect of change of "background" under the influence of the oscillation, $\chi_0 \neq X$ (only this effect is taken into account in the quasilinear method). The second term is related to the effects of coherence. The third term describes the "proper" interaction of the wave (only this effect is taken into account in the methods of Poincaré and Van Der Pol in their theory of oscillations).

Depending upon the form of the actual system, this or that nonlinear effect may be predominant; in the general case in finding the steady solution it is necessary to take into account all three effects.

**10.** Here we shall introduce two examples of finding steady solutions.

Let us consider Equation

$$-(x^{\cdot\cdot} + x) + x^{\cdot}(h + ax) + b(x^{\cdot})^2 = 0 \tag{10.1}$$

The equilibrium solution is $X = 0$; after linearizing with respect to the perturbation $x_1 e^{i\omega t}$ we obtain

$$D \equiv \omega^2 - 1 + i\omega h = 0$$

Hence

$$\gamma = {}^1/_2 h, \qquad \Omega = \sqrt{1 - h^2}$$

The critical value of the parameter is $h = 0$; the equilibrium is unstable when $h > 0$. To an accuracy of terms $\sim h$ we have $\Omega = 1$.

Let us seek the solution in the form

$$x = x_0 + x_{\sim} = x_0 + \sum_{\nu} x_\nu e^{i\nu\theta} \qquad (\nu \neq 0,\ \theta = \omega t)$$

After substituting for $x$ in Equation (10.1) we obtain

$$\Phi_0 \equiv -x_0 + 2b\omega^2 x_1 x_{-1} = 0 \tag{10.2}$$

$$\Phi_1 \equiv (\omega^2 - 1)x_1 + i\omega(hx_1 + ax_2x_{-1} + ax_0x_1) + 4b\omega^2 x_2 x_{-1} = 0 \tag{10.3}$$

$$\Phi_2 \equiv (4\omega^2 - 1)x_2 + i\omega(2hx_2 + ax_1^2) - b\omega^2 x_1^2 = 0 \tag{10.4}$$

In the coefficients in the nonlinear terms we can at once set $h = 0$, $\omega = 1$. From (10.2) and (10.4) we find that

$$\delta x = x_0 = 2bx_1x_{-1}, \ x_2 = {}^1/_3(b - ia)x_1^2;$$

substituting these expressions in (10.3), we obtain

$$\Phi_1 / x_1 \equiv D + q({}^1/_3 a^2 + {}^4/_3 b^2 + iab) = 0.$$

Hence, taking account of the fact that $\partial D/\partial\omega = 2$ and that $d\Omega/dh = 0$, when $h = 0$, we obtain $q = -\gamma / S^{(i)} = -h / ab$, $\omega = 1 - {}^1/_6 q(a^2 + b^2)$. The condition of existence of the solution is $ab < 0$.

Equation (10.1) after the substitution $x = \gamma y$ reduces to the form

$$y^{\cdot\cdot} + y = \gamma(2y^{\cdot} + ayy^{\cdot} + b(y^{\cdot})^2) \tag{10.5}$$

The equation of the form

$$y^{\cdot\cdot} + y = \gamma f(y, y^{\cdot}, \gamma) \tag{10.6}$$

where $\gamma$ is a small parameter, can be solved by the methods of Van Der Pol and Poincaré [7]. Setting $y = K \cos t$ we have according to the first method

$$K^{\cdot} = \gamma A(K), \qquad A(K) = -\frac{1}{2\pi}\int_0^{2\pi} f(K\cos u, -K\sin u, 0)\sin u\,du \tag{10.7}$$

Applying (10.7) to (10.5), we obtain

$$K^{\cdot} = \gamma K \tag{10.8}$$

whence follows the incorrect conclusion that Equation (10.5) has no steady solution except the equilibrium solution $K = 0$.

Equation (10.5) can be solved by the method of averaging [5]. For this we set $y^{\cdot} = z$ in Equation (10.5) and introduce the new unknowns $y = r\cos\varphi$, $z = r\sin\varphi$. Expressing $r^{\cdot}$ and $\varphi^{\cdot}$ in terms of $r$ and $\varphi$ and taking the ratio $r^{\cdot}/\varphi^{\cdot}$, we obtain an equation of the form $dr/d\varphi = \gamma f^*(r, \varphi, \gamma)$; Solving this by the method of averaging, we obtain, in the second approximation $dr/d\varphi = \gamma r(1 + {}^1/_2\gamma abr^2)$ with respect to $\gamma$.

The steady solution exists if $ab < 0$, and has the form $r^2 = -2/\gamma ab$; if we return to the variable $x$ and write down the solution in complex form, then we arrive at the expression for $q$ obtained above.

The example considered is typical in the theory of oscillations. The simplest systems in which there exist steady oscillations which are almost harmonic, are described by Equations [7]

$$x^{\cdot\cdot} = g(x, x^{\cdot}) \tag{10.9}$$

This equation can be reduced to the form (10.6). The equilibrium solution of Equation (10.9) is found from Equation $g(\chi, 0) = 0$. We shall assume that $\chi = 0$ (this can always be achieved by the substitution $x = \chi + x^*$). Assuming that the amplitude of the oscillation is small, let us expand (10.9) in a series with respect to $x$ and $x^{\cdot}$

$$x^{\cdot\cdot} - \left(\frac{\partial g}{\partial x}\right)x = \left(\frac{\partial g}{\partial x^{\cdot}}\right)x^{\cdot} + \sum_{n>1}\frac{d^n g}{n!}, \qquad d^n g = \left(x\frac{\partial}{\partial x} + x^{\cdot}\frac{\partial}{\partial x^{\cdot}}\right)^n g \tag{10.10}$$

Here the derivatives of $g$ are taken when $x = x^{\cdot} = 0$. Perturbations of the equilibrium state have the character of oscillations if for $\lambda = \lambda_*$ (when $\partial g/\partial x^{\cdot} \equiv 2\gamma = 0$) the inequality $\partial g/\partial x \equiv -\omega^2 < 0$ is satisfied. We shall assume that $\omega = 1$ (this can always be achieved by the substitution $t = \omega^{-1}t^*$); after the substitution $x = \gamma y$ we obtain from (10.10) (the increment $\gamma$ is assumed small)

$$y^{\cdot\cdot} + y = \gamma(2y^{\cdot} + g_2) + \sum_{n>2}\gamma^{n-1}g_n, \qquad g_n = \frac{1}{n!}\left(y\frac{\partial}{\partial x} + y^{\cdot}\frac{\partial}{\partial x^{\cdot}}\right)g \tag{10.11}$$

The abbreviated equation (10.7) for the starting equation (10.11) has the form (10.8), since the quadratic terms appearing in $g_2$ do not give contributions to the expression $A(K)$.

In order to solve Equation (10.11) by the method of averaging, it is necessary to revert to the second (or higher) approximation (the equation of the first approximation coincides with the equation obtained by the method of Van Der Pol, and has the form (10.8)).

When Equation (10.9) does not change on replacing $x$ by $(-x)$, i.e. when $g(-x, -x^{\cdot}) = -g(x, x^{\cdot})$, the terms with even $n$ are missing from the expansion (10.10); in this particular case we can make the substitution $x = \sqrt{\gamma}y$ and obtain

$$y^{\cdot\cdot} + y = \gamma(2y^{\cdot} + g_3) + \sum_{k>1}\gamma^k g_{2k+1} \tag{10.12}$$

With the help of (10.7) we find the steady amplitude

$$K^2 = -16\left(\frac{\partial^3 g}{\partial x^2 \partial x^{\cdot}} + \frac{\partial^3 g}{\partial (x^{\cdot})^3}\right)^{-1}$$

if the expression in brackets is negative.

Equations (10.9) not changing with the substitution of $x$ by $(-x)$ are remarkable in that their periodic solutions (if they exist) do not contain the even harmonics. Here we show only that the even harmonics are absent from oscillations of sufficiently small amplitude, when the expansion (10.10) is valid (not containing terms with even $n$ in the specified case). Let us substitute in (10.10) the series

$$x = \sum_{\mu=-\infty}^{\infty} x_\mu e^{i\mu\theta}$$

Then the differential $d^n g$ gives a contribution to the expression $\Phi_\nu$ in the form of the terms $cx_{\nu_1} \ldots x_{\nu n}$, where $\nu_1 + \ldots + \nu_n = \nu$, and $c$ is a certain coefficient depending on $\omega$ and $\lambda$, but not on $x$. If $\nu$ is even but $n$ is odd, then among the members $\nu_1, \ldots, \nu_n$ is at least one even number. Therefore for even values of $\nu$ the equations $\Phi_\nu = 0$ will be rigorously satisfied if we set all the even harmonics equal to zero.

Let us consider the equation (*)

$$x^{\cdot\cdot} + x = x^{\cdot}(h - x^2) \tag{10.13}$$

The dispersion relation $D = 0$ was considered in the first example. We shall seek the solution in the form

$$x = x_0 + \sum_\nu x_\nu e^{i\nu\theta} \qquad (\nu \neq 0, \ \theta = \omega t)$$

In Equation (10.13) there are no quadratic terms in $x$, therefore from the equations $\Phi_0 = 0$, $\Phi_2 = 0$ we find $\delta x = x_0 = 0, x_2 = 0$. For $\Phi_1$ we have $\Phi_1 \equiv x_1(D - i\omega q) = 0$ $(q = x_1 x_{-1})$.
Hence

$$P = (-i\omega)_{h=0} = -i, \qquad S = -{}^1/_2\, i, \qquad q = h, \qquad \omega = 1, \qquad \delta\omega = 0$$

For the calculation of higher harmonics $x_\nu$ we need to construct the expression for $\Phi_\nu$ in accordance with Equations (2.2) and (3.1). Integration of the linear terms is easily accomplished.

Let us first write down the nonlinear term in the form $x^2 x^{\cdot} = {}^1/_3\,(x^3)^{\cdot}$; the integral

$$I_{n\nu} = \int_0^{2\pi} x^n e^{-i\nu\theta}\,\frac{d\theta}{2\pi} \qquad \left(x = \sum_\mu x_\mu e^{i\mu\theta}\right)$$

is calculated after first of all raising the series for $x$ to the degree according to the binomial rule

$$I_{n\nu} = n!\Sigma \qquad \Sigma = \sum \frac{(x_{\nu_1})^{n_{\nu_1}}}{(n_{\nu_1})!} \cdots \frac{(x_{\nu_s})^{n_{\nu_s}}}{(n_{\nu_s})!} \tag{10.14}$$

Here the sum is taken over all the numbers $\nu$ (among which none are the same) and the positive numbers $n_\nu$, satisfying the conditions

$$n_{\nu_1} + \ldots + n_{\nu_s} = n, \ \nu_1 n_{\nu_1} + \ldots + \nu_s n_{\nu_s} = n$$

In the case under consideration $n = 3$. Now we have

$$\Phi_\nu \equiv (\omega^2\nu^2 - 1)\,x_\nu + i\omega\nu\,(hx_\nu - 2\Sigma') = 0$$

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*) The substitution $x = \sqrt{h}x^*$ reduces this equation to the equation of Van Der Pol.

where $\Sigma'$ differs from the sum in (10.14) in that the numbers $\nu$, according to (5.1), satisfy the supplementary condition $1 \leqslant \nu_1, \ldots, \nu_s < \nu$. The coefficients in the expressions $\Phi_\nu$ must be taken when $\lambda = \lambda_*$ (in the given case when $h = 0$, $\omega = 1$); therefore

$$x_\nu = \Sigma' \, 2i\nu \, / \, (\nu^2 - 1) \tag{10.15}$$

Equation (10.13) is not changed by the substitution of $x$ by $(-x)$ and therefore we can at once set the even harmonics equal to zero. For the odd harmonics we obtain from (10.14) and (10.15)

$$x_3 = {}^1/_8 i x_1^3, \qquad x_5 = {}^5/_{24} i x_1^2 x_3$$
$$x_7 = {}^7/_{24}\, i\, (x_1 x_3^2 + x_5 x_1^2)$$
$$x_9 = {}^9/_{40} i\, ({}^1/_6 x_3^3 + x_1 x_3 x_5 + {}^1/_2 x_1^2 x_7)$$

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